

Nomura algebras of nonsymmetric Hadamard models

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- nonsymmetric Hadamard model \in spin model.
- spin model (Jones in 1989)
→ an invariant of knots and links.
- In 1997, a spin model \subset association schemes,
a spin model \in the Nomura algebra.
- A spin model is one of the combinatorial objects.
- *We are interested in association schemes with a spin model.*

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Motivation 1

- W : a spin model
(a square matrix)
(type II+type III) $\xrightarrow{\text{construction}}$ “algebra”
attached to a spin model.
 $\mathcal{N}(W) = \text{“Nomura algebra”}$
- $\mathcal{N}(W)$ = the Bose-Mesner algebra
of an association scheme
(Jaeger-Matsumoto-Nomura in 1997)
- W : given \longrightarrow Determine $\mathcal{N}(W)$ by describing
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Motivation 2

Hadamard model W

(Nomura in 1994)

$$\left[\begin{array}{cc|cc} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \hline \omega H^T & -\omega H^T & A & A \\ -\omega H^T & \omega H^T & A & A \end{array} \right]$$

A : a Potts model,
 H : a Hadamard matrix,
 $\omega^4 = 1$.

Hadamard graph.

$\mathcal{N}(W) = \langle A_i \mid i = 0, \dots, 4 \rangle$,
 $\{A_i\}_{i=0}^4$: adj.matrices
of the Hadamard graph.

nonsymmetric

Hadamard model W'

(Jaeger-Nomura in 1999)

$$\left[\begin{array}{cc|cc} A & A & \xi H & -\xi H \\ A & A & -\xi H & \xi H \\ \hline -\xi H^T & \xi H^T & A & A \\ \xi H^T & -\xi H^T & A & A \end{array} \right]$$

A : a Potts model,
 H : a Hadamard matrix,
 ξ = a primitive 8-th root
of unity

→ ?
→ $\mathcal{N}(W') = ?$

$X =$ a nonempty finite set, $n = |X|$,
 $\text{Mat}_X(\mathbb{C}) =$ the full matrix ring on \mathbb{C} ,
 $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Definition:

- (1) Hadamard matrix $H \iff HH^T = nI, H_{ij} = \pm 1$,
 (2) complex Hadamard matrix $C \iff C\bar{C}^T = nI, |C_{ij}| = 1$,
 (3) W : a type II matrix $\iff W(W_-)^T = nI$,
 $(W_-)_{ij} = \frac{1}{W_{ij}}$.

Example of (3):

a Potts model: $A = u^3I - u^{-1}(J - I)$ with $(u^2 + u^{-2})^2 = n$.

Let $W \in \text{Mat}_X(\mathbb{C}^*)$ be a **type II** matrix.

$Y_{ij} \in \mathbb{C}^n$: a column vector s.t.

$$Y_{ij}(x) \stackrel{\text{def}}{=} \frac{W_{x,i}}{W_{x,j}} \quad \text{for } \forall i, j \in X,$$

$\mathcal{N}(W) \stackrel{\text{def}}{=} \{A \in \text{Mat}_X(\mathbb{C}^*) \mid AY_{ij} = \theta_{ij}Y_{ij}, \forall i, j \in X\}$.

Theorem (Jaeger-Matsumoto-Nomura, 1997)

Let $W \in \text{Mat}_X(\mathbb{C}^*)$ be a **type II** matrix.

Then, $\mathcal{N}(W)$ is the Bose–Mesner algebra of an association scheme.

W = a **type III** matrix

$$\stackrel{\text{def}}{\iff} \sum_{x \in X} \frac{W_{a,x} W_{b,x}}{W_{c,x}} = \sqrt{n} \frac{W_{a,b}}{W_{a,c} W_{c,b}} \text{ for } \forall a, b, c \in X.$$

If $W \in \text{Mat}_X(\mathbb{C}^*)$: a **type II**, a **type III**,
then W is called a **spin model**.

Example: **Hadamard model**

$$W = \left[\begin{array}{cc|cc} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \hline \omega H^T & -\omega H^T & A & A \\ -\omega H^T & \omega H^T & A & A \end{array} \right],$$

A : a Potts model, H : a Hadamard matrix,
 $\omega^4 = 1$.

Theorem (Jaeger-Matsumoto-Nomura, 1997)

Let $W \in \text{Mat}_X(\mathbb{C}^*)$ be a **spin model**.

Then, $W \in \mathcal{N}(W)$.

Hadamard graph

The adjacency matrices of **Hadamard graph** are the following:

$$A_0 = I_{4n},$$

$$A_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} J-I & J-I & 0 & 0 \\ J-I & J-I & 0 & 0 \\ 0 & 0 & J-I & J-I \\ 0 & 0 & J-I & J-I \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}.$$

Graph description:

$G = (X \times X, \sim)$: an undirected graph on the vertex set $X \times X$,
 $(a, b) \sim (c, d) \stackrel{\text{def}}{\iff} \langle Y_{ab}, Y_{cd} \rangle \neq 0$ (Hermitian inner product).

For $C \subset X \times X$, we denote by $A(C)$ the matrix in $\text{Mat}_X(\mathbb{C}^*)$ with (a, b) -entry equal to 1 if $(a, b) \in C$ and to 0 otherwise.

Theorem (Jaeger-Matsumoto-Nomura, 1997)

Let C_1, \dots, C_p be the connected components of G .

Then, the algebra $\mathcal{N}(W)$ has a basis $\{A(C_i) \mid i = 1, \dots, p\}$.

Theorem (Jaeger-Matsumoto-Nomura, 1997)

Let $W \in \text{Mat}_X(\mathbb{C}^*)$ be a *Hadamard model*.

Then, $\mathcal{N}(W) = \langle A_i \mid i = 0, \dots, 4 \rangle$,

where $\{A_i\}_{i=0}^4$ are the adjacency matrices of *Hadamard graph*.

$\langle A_i \mid i = 0, \dots, 4 \rangle \subset \mathcal{N}(W)$: easy.

$\langle A_i \mid i = 0, \dots, 4 \rangle \supset \mathcal{N}(W)$: difficult!

triple intersection numbers,
quadruple intersection numbers.

nonsymmetric Hadamard model (Jaeger-Nomura, 1999)

nonsymmetric Hadamard model W' :

$$W' = \begin{bmatrix} A & A & \xi H & -\xi H \\ A & A & -\xi H & \xi H \\ -\xi H^T & \xi H^T & A & A \\ \xi H^T & -\xi H^T & A & A \end{bmatrix}$$

We want to determine $\mathcal{N}(W')$.

directed Hadamard graph (Jaeger-Nomura, 1999)

We define the adjacency matrices $\{A'_i\}_{i=0}^4$ of the **directed Hadamard graph**:

$$A'_0 = A_0,$$

$$A'_2 = A_2,$$

$$A'_4 = A_4,$$

$$A'_1 = \left[\begin{array}{cc|cc} 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ \hline \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \end{array} \right],$$

$$A'_3 = \left[\begin{array}{cc|cc} 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ \hline \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \end{array} \right].$$

We define

$$A_i^+ = \begin{bmatrix} I_{2n} & 0 \\ 0 & 0 \end{bmatrix} A_i, \quad A_i^- = \begin{bmatrix} 0 & 0 \\ 0 & I_{2n} \end{bmatrix} A_i.$$

Then

$$\langle A_i^\pm \mid i = 0, \dots, 4 \rangle$$

is a coherent algebra with automorphism ρ :

$$\rho : A_1^+ \leftrightarrow A_3^-, A_1^- \leftrightarrow A_3^+, A_i^+ \leftrightarrow A_i^- \quad (i = 0, 2, 4).$$

Klin, Muzychuk, Pech, Woldar, and Zieschang in 2007:

$$\mathcal{A}' = \langle A'_1 = A_1^+ + A_3^-, A'_3 = A_1^- + A_3^+, A_0, A_2, A_4 \rangle$$

is a coherent algebra. Indeed, the Bose–Mesner algebra of the directed Hadamard graph.

$$W = \begin{bmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega H^T & -\omega H^T & A & A \\ -\omega H^T & \omega H^T & A & A \end{bmatrix}, W' = \begin{bmatrix} A & A & \xi H & -\xi H \\ A & A & -\xi H & \xi H \\ -\xi H^T & \xi H^T & A & A \\ \xi H^T & -\xi H^T & A & A \end{bmatrix}.$$

Theorem (Munemasa-I.)

$$\mathcal{N}(W') = \langle A'_i \mid i = 0, \dots, 4 \rangle.$$

Proof.

G : Graph description of $\mathcal{N}(W)$,
 (Jaeger-Matsumoto-Nomura:
 triple, quadruple intersection numbers)

G' : Graph description of $\mathcal{N}(W')$.

$\exists \sigma$: a permutation of $X \times X$
 giving an isomorphism $G \rightarrow G'$,
 $A_i \mapsto A'_i$.