Connectivity of the space of spherical designs

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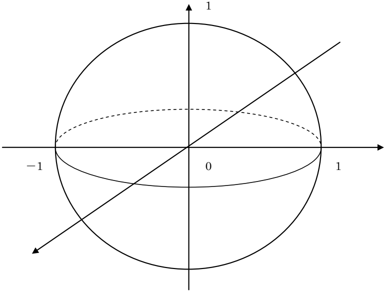
- $d \geq 2$
- $S^{d-1} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1^2 + \ldots + x_d^2 = 1\}$ (unit sphere),
- $B^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1^2 + \ldots + x_d^2 \leq 1\}$ (unit ball).

Example:

$S^2 = \{(x_1, x_2, x_3) \in \boldsymbol{R}^3 | {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$,

$B^3 = \{(x_1, x_2, x_3) \in \boldsymbol{R}^3 | {x_1}^2 + {x_2}^2 + {x_3}^2 \leq 1\}$.



Definition (Delsarte-Goethals-Seidel, 1977)

$X \subset S^{d-1}$, X: a non-empty finite set,
X: a spherical t-design

$$\iff \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)dw(x) = \frac{1}{|X|}\sum_{x\in X} f(x)$$

for $\forall f(x)$ with $\deg f \leq t$.

$t = 1 \implies$ the center of gravity of $X =$ the origin.

Seymour-Zaslavsky in 1984
$\longrightarrow \exists$ a spherical t-design $\subset S^{d-1}$ for $\forall d, t$.

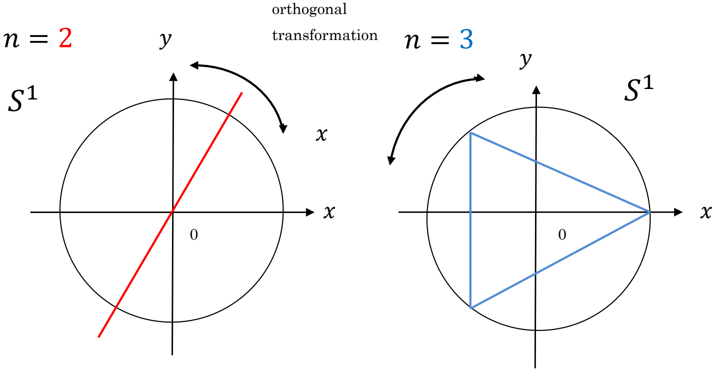
Theorem (Hong, 1982)

$X \subset S^1$, $n = |X|$.
X: a spherical t-design $\Longrightarrow$

(1) $t + 1 \leq n \leq 2t + 1 \Longrightarrow X$*: a regular n-gon,*
(2) $n = 2t + 2 \Longrightarrow X$*: the union of two regular $(t+1)$-gons,*
(3) $\forall n \geq 2t + 3 \Longrightarrow \exists\ (\aleph_1) X$*'s which cannot be decomposed into the union of degree k_i-gons where $k_i \geq (t+1)$.*

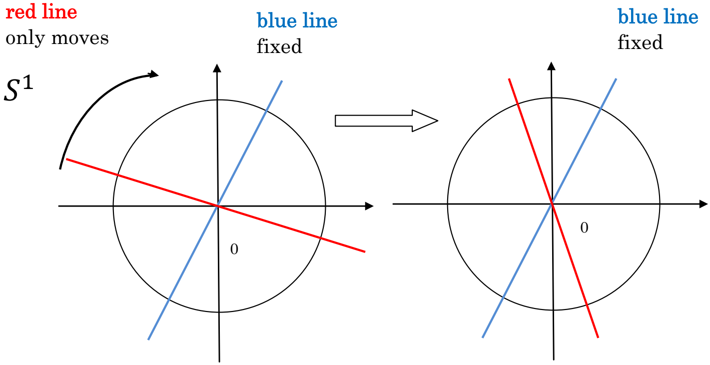
(1) $t+1 \leq n \leq 2t+1 \ \rightarrow$ regular $n-$gon

$t=1$ in Hong's theorem $\Rightarrow \ n = 2, 3$



(2) $n = 2t + 2$ $\rightarrow$ union of regular $(t+1)-$gons

$t = 1$ in Hong's theorem $\Rightarrow n = 4$



Definition (Bannai)

$X = \{x_1, \ldots, x_n\} \subset S^{d-1}$: a spherical t-design.

X: non-rigid $\iff$

$\forall \epsilon > 0$, $\exists$ a spherical t-design $X' = \{x'_1, \ldots, x'_n\} \subset S^{d-1}$ s.t.

(1) $||x_i - x'_i|| < \epsilon$ for $\forall i \in \{1, \ldots, n\}$,

(2) $\nexists T \in O(d, \mathbb{R})$ s.t. $Tx_i = x'_i$ for $\forall i \in \{1, \ldots, n\}$.

X: rigid if X is not non-rigid.

Conj. A: For $\forall\, t$ and d, if $|X|$ is sufficiently large (i.e., greater than a certain number $f(t,d)$), then X is non-rigid.

Conj. B: For each fixed pair of t and d, $\exists$ only finitely many rigid spherical t-designs up to orthogonal transformations.

Conj. A $\Longleftrightarrow$ Conj. B.

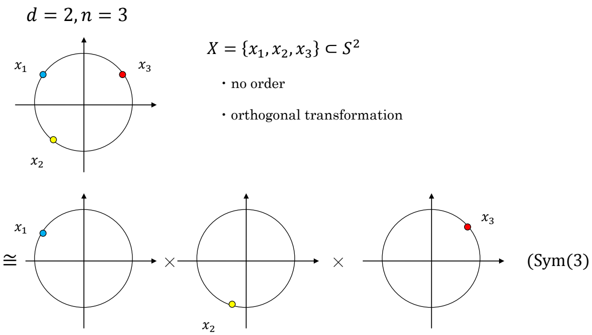
Conj. B $\Longrightarrow$ Conj. A: trivial,

Conj. A $\Longrightarrow$ Conj. B: by Lyubich and Vaserstein in 1993.

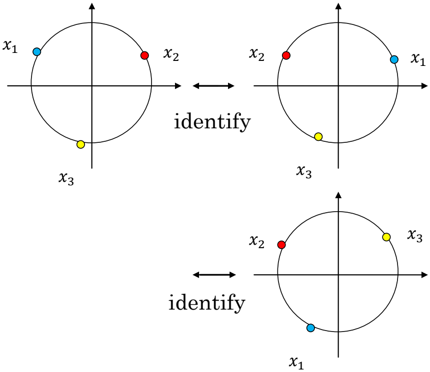
Theorem (Bannai, 1987)

If X is a rigid spherical t-design in S^1, then X consists of $k+1$ vertices of a regular $(k+1)$-gon with $t \leq k \leq 2t$.

$X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\} \subset S^{d-1}$: a spherical t-design



Fix an ordering of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$,
$X = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in (S^{d-1})^n$.
topological space



$\{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$: distinct$\}/(\mathrm{Sym}(n) \times O(d, \mathbb{R}))$
$\longleftrightarrow$ permutation of coordinates, rotation.

$D_d(n, t) = \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in (S^{d-1})^n \mid (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$: a spherical t-design$\}$.

the set of spherical t-design with n (possibly repeated) points in S^{d-1}.

$D_d(n,t) = \{(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n) \in (S^{d-1})^n \mid (\boldsymbol{x}_1,\dots,\boldsymbol{x}_n) : \text{a spherical } t\text{-design}\}$,
$\overline{D}_d(n,t) = D_d(n,t)/\mathrm{Sym}(n)$.

We denote $[\boldsymbol{x}_1,\dots,\boldsymbol{x}_n] \in \overline{D}_d(n,t)$.
For $T \in O(d,\mathbb{R})$,

$$T[\boldsymbol{x}_1,\dots,\boldsymbol{x}_n] = [T\boldsymbol{x}_1,\dots,T\boldsymbol{x}_n],$$

$$[[\boldsymbol{x}_1,\dots,\boldsymbol{x}_n]] = \{T[\boldsymbol{x}_1,\dots,\boldsymbol{x}_n] \mid T \in O(d,\mathbb{R})\},$$
$$\mathcal{D}_d(n,t) = \{[[\boldsymbol{x}_1,\dots,\boldsymbol{x}_n]] \mid [\boldsymbol{x}_1,\dots,\boldsymbol{x}_n] \in \overline{D}_d(n,t)\}.$$

Lemma

$X \in D_d(n,t)$: *rigid* $\Longleftrightarrow$ $[[X]]$ *is an isolated point in* $\mathcal{D}_d(n,t)$.

In what follows, we only consider the case of $t = 1$, i.e., spherical 1-design $\in (S^{d-1})^n$.

$d \geq 2$.
$\boldsymbol{a} \in B^d$, $\boldsymbol{u} \in S^{d-1}$.

$$D_d(n,t) = \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in (S^{d-1})^n \mid (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) : \text{a spherical } t\text{-design}\}.$$

$$\Omega_d(n,\boldsymbol{a},\boldsymbol{u}) := \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in (S^{d-1})^n \mid \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i = \boldsymbol{a},\ \boldsymbol{x}_i = \boldsymbol{u} \text{ for some } i\}.$$

$$\begin{aligned}\Omega_d(n,\boldsymbol{a}) &:= \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in (S^{d-1})^n \mid \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i = \boldsymbol{a}\} \\ &= \bigcup_{\boldsymbol{u} \in S^{d-1}} \Omega_d(n,\boldsymbol{a},\boldsymbol{u}).\end{aligned}$$

$\Omega_d(n,\boldsymbol{o}) = D_d(n,1)$.
$\Omega_d(n,\boldsymbol{a})$: a metric induced from the Euclidean metric in $\mathbb{R}^{dn}$
$\longrightarrow \Omega_d(n,\boldsymbol{a})$: a topological space.

$\overline{\Omega}_d(n, \boldsymbol{a}) = \Omega_d(n, \boldsymbol{a})/\mathrm{Sym}(n)$.

Lemma

The topological space $\overline{\Omega}_d(n, \boldsymbol{a}) \neq \emptyset \iff n \geq 2$ *or* $\boldsymbol{a} \in S^{d-1}$.
$\overline{\Omega}_d(n, \boldsymbol{a}) \neq \emptyset \implies \overline{\Omega}_d(n, \boldsymbol{a})$: *connected.*

We recall

$$D_d(n, t) = \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in (S^{d-1})^n \mid (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) : \text{a spherical } t\text{-design}\},$$
$$\overline{D}_d(n, t) = D_d(n, t)/\mathrm{Sym}(n).$$

$$\implies \overline{\Omega}_d(n, \boldsymbol{o}) = \overline{D}_d(n, 1).$$

Theorem (Y. Inui, A. Munemasa and T.I.)

The topological space $\mathcal{D}_d(n, 1)$ *is connected if* $\mathcal{D}_d(n, 1) \neq \emptyset$.

$n \geq 4$,
$n = (n-2) + 2$.
$\exists$ a spherical 1-design $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-2}) \in D_d(n-2, 1)$ of $n-2$ points in S^{d-1}. Let $\boldsymbol{e}_1, \boldsymbol{e}_2 \in S^{d-1}$ be orthogonal, and set

$$\boldsymbol{y}_\theta = \cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2,$$
$$Y_\theta = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-2}, \boldsymbol{y}_\theta, -\boldsymbol{y}_\theta).$$

Then $Y_\theta \in D_d(n, 1)$, and

$$|\{[[Y_\theta]] \mid 0 \leq \theta \leq \pi\}| > 1.$$

$$\longrightarrow |\mathcal{D}_d(n, 1)| > 1.$$

$\mathcal{D}_d(n, 1)$: connected $\longrightarrow$ $\mathcal{D}_d(n, 1)$ has no isolated points.
Therefore, $\nexists$ rigid spherical 1-designs of n points.

Corollary (Y. Inui, A. Munemasa and T.I.)

Let $X \in D_d(n, 1)$ be a spherical 1-design. Then the following are equivalent.

1. *X is a rigid spherical 1-design,*
2. *$n = 2$ or 3, and X consists of a pair of antipodal points, or X consists of the vertices of an equilateral triangle.*