Nomura algebra of nonsymmetric Hadamard models

Takuya Ikuta
(joint work with Akihiro Munemasa)

Kobe Gakuin University

Geometric and Algebraic Combinatorics 5.
Oisterwijk, The Netherlands, August 18, 2011.

Motivation

W: a spin model $\longrightarrow$ "algebra" attached to a spin model

type II $+$ type III

"algebra"$=$ Nomura algebra $\mathcal{N}(W)$

$\mathcal{N}(W) =$ the Bose-Mesner algebra of an association scheme

(Jaeger-Matsumoto-Nomura in 1997)

W: given $\longrightarrow$ Determine $\mathcal{N}(W)$ by describing the association scheme

Hadamard model W

Nomura in 1994

$$\begin{bmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega H^T & -\omega H^T & A & A \\ -\omega H^T & \omega H^T & A & A \end{bmatrix}$$

A: a Potts model,
H: a Hadamard matrix,
$\omega^4 = 1$.

nonsymmetric Hadamard model W'

Jaeger-Nomura in 1999

$$\begin{bmatrix} A & A & \xi H & -\xi H \\ A & A & -\xi H & \xi H \\ -\xi H^T & \xi H^T & A & A \\ \xi H^T & -\xi H^T & A & A \end{bmatrix}$$

$\xi =$ a primitive 8-th root of unity

Hadamard graph. $\longrightarrow$ **?**

$\mathcal{N}(W) = \langle A_i \mid i = 0, \ldots, 4 \rangle$, $\longrightarrow$ $\mathcal{N}(W') =$ **?**

$\{A_i\}_{i=0}^4$: adj.matrices of the Hadamard graph.

X = a nonempty finite set, $n = |X|$,
$\mathrm{Mat}_X(\mathbb{C}^*)$ = the full matrix ring on $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Definition:

(1) Hadamard matrix H $\longleftrightarrow$ $HH^T = nI, \ H_{ij} = \pm 1$,

(2) complex Hadamard matrix C $\longleftrightarrow$ $C\overline{C}^T = nI, \ |C_{ij}| = 1$,

(3) a type II matrix W $\longleftrightarrow$ $W(W_-)^T = nI$,

$$(W_-)_{ij} = \frac{1}{W_{ij}}.$$

Example of (3):
a Potts model: $A = u^3 I - u^{-1}(J - I) \in \mathrm{Mat}_X(\mathbb{C}^*)$ with $(u^2 + u^{-2})^2 = n$.

Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ be a type II matrix,

$Y_{ij} \in \mathbb{C}^n$: a column vector s.t.

$$Y_{ij}(x) \overset{\text{def}}{=} \frac{W_{x,i}}{W_{x,j}} \quad \text{for} \quad \forall i,j \in X,$$

$$\mathcal{N}(W) \overset{\text{def}}{=} \{A \in \mathrm{Mat}_X(\mathbb{C}^*) \mid AY_{ij} = \theta_{ij} Y_{ij},\ \forall i,j \in X\}.$$

Theorem (Jaeger-Matsumoto-Nomura, 1997)

*Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ be a type II matrix.
Then, $\mathcal{N}(W)$ is the Bose–Mesner algebra of an association scheme.*

$\mathcal{N}(W) =$ Nomura algebra.

W = a type III matrix

$$\overset{\text{def}}{\iff} \sum_{x\in X} \frac{W_{a,x}W_{b,x}}{W_{c,x}} = D\frac{W_{a,b}}{W_{a,c}W_{c,b}} \quad \text{for } \forall a,b,c \in X.$$

If $W \in \mathrm{Mat}_X(\mathbb{C}^*)$: a type II, a type III,
then W is called a spin model.

Example: Hadamard model

$$W = \begin{bmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega H^T & -\omega H^T & A & A \\ -\omega H^T & \omega H^T & A & A \end{bmatrix},$$

A: a Potts model, H: a Hadamard matrix,
$\omega^4 = 1$.

Theorem (Jaeger-Matsumoto-Nomura, 1997)

Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ be a spin model.
Then, $W \in \mathcal{N}(W)$.

Hadamard graph

The adjacency matrices of Hadamard graph are the following:

$$A_0 = I_{4n},$$

$$A_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} J-I & J-I & 0 & 0 \\ J-I & J-I & 0 & 0 \\ 0 & 0 & J-I & J-I \\ 0 & 0 & J-I & J-I \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}.$$

Graph description:
$G = (X \times X, \sim)$: an undirected graph on the vertex set $X \times X$,
$(a,b) \sim (c,d) \overset{\text{def}}{\iff} \langle Y_{ab}, Y_{cd} \rangle \neq 0$ (Hermitian inner product).

For $C \subset X \times X$, we denote by $A(C)$ the matrix in $\mathrm{Mat}_X(\mathbb{C}^*)$ with (a,b)-entry equal to 1 if $(a,b) \in C$ and to 0 otherwise.

Theorem (Jaeger-Matsumoto-Nomura, 1997)

Let $C_1, \ldots, C_p$ be the connected components of G.
Then, the algebra $\mathcal{N}(W)$ has a basis $\{A(C_i) \mid i = 1, \ldots, p\}$.

Theorem (Jaeger-Matsumoto-Nomura, 1997)

Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ be a Hadamard model.
Then, $\mathcal{N}(W) = \langle A_i \mid i = 0, \ldots, 4 \rangle$,
where $\{A_i\}_{i=0}^4$ are the adjacency matrices of Hadamard graph.

triple intersection numbers, quadruple intersection numbers.

nonsymmetric Hadamard model (Jaeger-Nomura, 1999)

nonsymmetric Hadamard model W':

$$W' = \begin{bmatrix} A & A & \xi H & -\xi H \\ A & A & -\xi H & \xi H \\ -\xi H^T & \xi H^T & A & A \\ \xi H^T & -\xi H^T & A & A \end{bmatrix}$$

We want to determine $\mathcal{N}(W')$.

We define the adjacency matrices $\{A'_i\}_{i=0}^4$ of the directed Hadamard graph:

$$A'_0 = A_0,$$
$$A'_2 = A_2,$$
$$A'_4 = A_4,$$
$$A'_1 = \begin{bmatrix} 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \end{bmatrix},$$
$$A'_3 = \begin{bmatrix} 0 & 0 & \frac{1}{2}(J-H) & \frac{1}{2}(J+H) \\ 0 & 0 & \frac{1}{2}(J+H) & \frac{1}{2}(J-H) \\ \frac{1}{2}(J+H^T) & \frac{1}{2}(J-H^T) & 0 & 0 \\ \frac{1}{2}(J-H^T) & \frac{1}{2}(J+H^T) & 0 & 0 \end{bmatrix}.$$

We define

$$A_i{}^+ = \begin{bmatrix} I_{2n} & 0 \\ 0 & 0 \end{bmatrix} A_i, \quad A_i{}^- = \begin{bmatrix} 0 & 0 \\ 0 & I_{2n} \end{bmatrix} A_i.$$

Then

$$\langle A_i{}^\pm \mid i = 0, \dots, 4 \rangle$$

is a coherent algebra with automorphism ρ:

$$\rho : A_1{}^+ \leftrightarrow A_3{}^-,\ A_1{}^- \leftrightarrow A_3{}^+,\ A_i{}^+ \leftrightarrow A_i{}^- \ (i = 0, 2, 4).$$

Klin, Muzychuk, Pech, Woldar, and Zieschang in 2007:

$$\mathcal{A}' = \langle A_1' = A_1{}^+ + A_3{}^-,\ A_3' = A_1{}^- + A_3{}^+,\ A_0, A_2, A_4 \rangle$$

is a coherent algebra. Indeed, the Bose–Mesner algebra of the directed Hadamard graph.

$$W = \begin{bmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega H^T & -\omega H^T & A & A \\ -\omega H^T & \omega H^T & A & A \end{bmatrix}, W' = \begin{bmatrix} A & A & \xi H & -\xi H \\ A & A & -\xi H & \xi H \\ -\xi H^T & \xi H^T & A & A \\ \xi H^T & -\xi H^T & A & A \end{bmatrix}.$$

Theorem (Munemasa-I.)

$\mathcal{N}(W') = \langle A'_i \mid i = 0, \ldots, 4 \rangle.$

Proof.

G: Graph description of $\mathcal{N}(W)$,
(Jaeger-Matsumoto-Nomura:
triple, quadruple intersection numbers)
G': Graph description of $\mathcal{N}(W')$.
$\exists \sigma$: a permutation of $X \times X$
giving an isomorphism $G \to G'$,
$A_i \mapsto A'_i$.