Spin models constructed from Hadamard matrices

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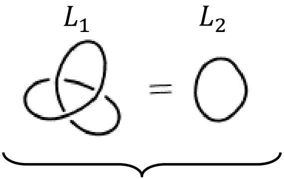
- a link $=$ a finite collection of mutually disjoint simple closed curves in $\mathbb{R}^3$.



$L_1 \approx L_2$

(ambient isotopy)

- a link diagram $=$ a projection of the link on a plane.
- Z : {link diagram} $\longrightarrow \mathbb{C}$.
- $L_1, L_2 \in$ {link diagram}, $L_1 \approx L_2 \Longrightarrow Z(L_1) = Z(L_2)$.

  Z : link invariant

$X =$ a finite set,
a **spin model** $\in \mathrm{Mat}_X(\mathbb{C}^*)$ $(\mathbb{C}^* = \mathbb{C} - \{0\})$,
$\exists$ a spin model $\Longrightarrow$ **link invariant.**

Who defined a spin model?

V.F.R. Jones in 1989, symmetric matrix.

spin model	$\subset$	generalized spin model	$\subset$	4-weight spin model
Jones		K. Kawagoe, A. Munemasa, Y. Watatani.		E. Bannai, E. Bannai.
$W^T = W$		$W^T \neq W$ or $W^T = W$		

In this talk, we treat only these parts.
"spin model"

- F. Jaeger, M. Matsumoto, and K. Nomura in 1998,
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$\Longrightarrow$ the structure theorem for spin models.

Definition of spin models

Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$.

- W : a type II matrix $\stackrel{\text{def}}{\iff} \forall \alpha, \beta \in X$:

$$\sum_{x \in X} \frac{W(\alpha, x)}{W(\beta, x)} = |X| \delta_{\alpha,\beta}.$$

- W : a type III matrix $\stackrel{\text{def}}{\iff} \forall \alpha, \beta, \gamma \in X$:

$$\sum_{x \in X} \frac{W(\alpha, x) W(\beta, x)}{W(\gamma, x)} = D \frac{W(\alpha, \beta)}{W(\alpha, \gamma) W(\gamma, \beta)},$$

  where $D^2 = |X|$.

- W = type II and type III $\stackrel{\text{def}}{\iff}$ W is called a spin model.
- W_1, W_2 : spin models $\Longrightarrow$ $W_1 \otimes W_2$: spin model.

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Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ be a matrix s.t.

$$
\begin{aligned}
W &= \begin{pmatrix} a & b & \cdots & \cdots & b \\ b & a & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a & b \\ b & \cdots & \cdots & b & a \end{pmatrix}, \quad a \neq 0,\ b \neq 0, \\
&= aI + b(J - I).
\end{aligned}
$$

$$W = aI + b(J - I),$$

- $|X| \geq 2$,
  - W : a type II matrix $\Longrightarrow$

$$\frac{b}{a} + \frac{a}{b} + (|X| - 2) = 0.$$

  - W : a type II and a type III matrix $\Longrightarrow$
    - $a = u^3,\ b = -u^{-1},\ D = u^2 + u^{-2},\ |X| = D^2,$
    - $W = u^3 I - u^{-1}(J - I)$ with $D = u^2 + u^{-2}$.
- $|X| = 1$,
  - $W = (u)$,
  - W : a type III matrix $\Longrightarrow u^4 = 1$.

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- Potts model

  $I =$ identity matrix $\in \mathrm{Mat}_X(\mathbb{C}^*)$,
  $J =$ all 1's matrix $\in \mathrm{Mat}_X(\mathbb{C}^*)$,
  $r = |X|$.

$$A_u = \begin{cases} u^3 I - u^{-1}(J - I) & \text{if } r = (u^2 + u^{-2})^2 \geq 2, \\ (u) & \text{if } r = u^4 = 1. \end{cases}$$

  $A_u \longleftrightarrow$ the Jones polynomial.

W : a spin model.

- F. Jaeger, M. Matsumoto, and K. Nomura in 1998 :
  $W \in$ the Bose-Mesner algebra
  for some self-dual association schemes
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Known spin models

- Potts model,
- Jaeger's Higman-Sims model,
- Spin models on finite abelian groups,
- Hadamard model,
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Known spin models

- Jaeger's Higman-Sims model (F. Jaeger, 1992)

$$
\begin{aligned}
W &= -\tau^5 I - \tau A + \tau^{-1}(J - I - A) \in \mathrm{Mat}_X(\mathbb{C}), \\
&\quad \tau^2 + \tau^{-2} = 3, \\
&\quad A = \text{adjacency matrix of Higman-Sims graph.}
\end{aligned}
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W : a symmetric spin model.

- Spin models on finite abelian groups
(E. Bannai, E. Bannai and F. Jaeger, 1997)

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Known spin models

- Non-symmetric Hadamard model
  (F. Jaeger and K. Nomura, 1999)

$$W = \begin{array}{c} \\ X_0 \\ X_1 \end{array} \begin{array}{cc} X_0 & X_1 \\ \left(\begin{array}{cc} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \xi H \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \xi H^T & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u \end{array} \right) \end{array},$$

where
- $A_u =$ a Potts model,
- $H =$ any Hadamard matrix,
- $\xi =$ a primitive 8-th root of unity.

Then W is a spin model of index 2.

The size of W is $4r$, where $r =$ the size of Hadamard matrix H.

Known spin models

- Hadamard model (K. Nomura, 1994)

$$W' = \begin{array}{c} \\ X_0 \\ X_1 \end{array} \begin{array}{c} X_0 \qquad\qquad\qquad\qquad X_1 \\ \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u & \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \omega H \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \omega H^T & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u \end{pmatrix} \end{array},$$

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In this talk, $\mathbb{Z}_m$ will denote the subset $\{0, 1, \dots, m-1\}$ of integers, not the set of residue classes of integers modulo m.

We define the following:

For $\forall\ i, j, \ell, \ell' \in \mathbb{Z}_m$

- $L((i, \ell), (j, \ell')) \stackrel{\text{def}}{=} -2m(\ell - \ell')(i - j)$,
- $\epsilon(i, j) \stackrel{\text{def}}{=} (i-j)^2 + m\big(i - j\big)$,
- $\delta(i, j) \stackrel{\text{def}}{=} (i - j)^2$.

Theorem(A. Munemasa-I.)

$m =$ even.
$X = \{(i, \ell, x) | i \in \mathbb{Z}_m, \ell \in \mathbb{Z}_m, x \in Y = \{1, \dots, r\}\}$, $|X| = m^2 r$,
A_u : a Potts model of size r,
$H \in \mathrm{Mat}_Y(\mathbb{C}^*)$: any Hadamard matrix of size r.
For $\forall\, i, j \in \mathbb{Z}_m$

$$V_{ij} \stackrel{\text{def}}{=} \begin{cases} A_u & \text{if } i - j = \text{even}, \\ H & \text{if } (i, j) \equiv (0, 1) \pmod 2, \\ H^T & \text{if } (i, j) \equiv (1, 0) \pmod 2. \end{cases}$$

Define $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ by

$$W(\alpha, \beta) = a^{L((i,\ell),(j,\ell')) + \epsilon(i,j)} V_{ij}(x, y)$$

for $\forall\, \alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$,
where a is a primitive $2m^2$-th root of unity.
Then W is a spin model of index m.

$W_{H,u,a} \overset{\text{def}}{=}$ a spin model W of even index m given in the previous Theorem.

If $m = 2$, then we recover the non-symmetric Hadamard models of index 2 of Jaeger-Nomura.

Why do we suppose even index ?

F. Jaeger and K. Nomura, 1999:
"**Symmetric Versus Non-Symmetric Spin Models for Link Invariants.**"
J. Algebraic Combin. **10** (1999), 241–278.

"The link invariant of spin models of odd index is guage equivalent to the link invariant of symmetric spin models.
One might expect to obtain new non-symmetric spin models in the case where m is a power of 2."

$W_{H,u,a}$: a spin model given in the previous Theorem of index $m = 2^s q$, $q =$ odd.

- For example,
  $s = 1, q = 3$ $(m = 6)$
  $\Longrightarrow$
  $W_{H,u,a} = W_{H,u,a'} \otimes W'$,
  - $W_{H,u,a'}$: a non-symmetric Hadamard model (index 2),
  - $a' =$ a primitive 8-th root of unity,
  - W' is a spin model of index 3.
- $m = 2^s$ $(q = 1) \Longrightarrow$ decomposable ?

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Theorem (A. Munemasa-I.)

$W_{H,u,a}$: *a spin model whose index is a power of* 2.
$r =$ *the order of the Hadamard matrix* H.
$r > 4 \Longrightarrow$
$W_{H,u,a}$ *cannot be decomposed into a tensor product of known spin models.*

- $r = 4 \Longrightarrow W_{H,u,a} = H \otimes W_{(1),u,a}$,

$$H = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

- $r = 1$, index $m \equiv 0 \pmod 4) \Longrightarrow$
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Define $W' \in \mathrm{Mat}_X(\mathbb{C}^*)$ by

$$W'(\alpha, \beta) = \eta^{(\ell - \ell')(i - j)} b^{\delta(i,j)} V_{ij}$$

for $\forall\, \alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$,
where η be a primitive m-th root of unity, b is an m^2-th root of unity.
Then W' is a symmetric spin model.

$W'_{H,u,\eta,b} \stackrel{\text{def}}{=}$ a spin model W' of size m^2r given in the previous Theorem.

If $m = 2$, then we recover the symmetric Hadamard models of Nomura.

We note that the decomposability and identification with known spin models are yet to be determined for the following cases.

1. $W_{H,u,a}$: $r = 1$, $m \equiv 2 \pmod 4$,
2. $W'_{H,u,b}$: $r = 1$,
3. $W_{H,u,a}$ and $W'_{H,u,b}$: $r = 2$,
4. $W_{H,u,a}$ and $W'_{H,u,b}$: $r > 4$ and m is not a power of 2.